HYDRODYNAMIC AND THERMAL INSTABILITY OF A STEADY CONVECTIVE FLOW

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Stability of the steady convective motion in a layer of viscous fluid contained between two vertical, parallel planes heated to different temperatures, was investigated in [1 to 4]. It was shown, that the resulting flow consisting of two, mutually opposing convective streams becomes hydrodynamically unstable under the monotonous perturbations at relatively small values of the Grasshof number.

A nontrivial generalization of this problem is met in the problem on stability of convective motion in a layer, arbitrarily orientated with respect to the force of gravity. We find that the temperature gradient in an inclined fluid layer has a vertical component. If this vertical gradient is directed downward (heating from below) and is sufficiently high, then the instability may occur even in the fluid at rest. When the convective motion takes place, we obviously have two physically distinct mechanisms of instability the steady state: (1) the hydrodynamic instability of two opposing convective flows and (2) the thermal (convective) instability of the fluid heated from below. Thermal instability occurs when the layer is nearly horizontal, this position corresponds to heating from below. When the layer is vertical or positioned at such an angle that it corresponds to heating from above, then the mechanism of the instability is hydrodynamic. Within the transitional range of the angles of inclination, both mechanisms are active.

Below we study the stability of a steady convective flow in an inclined layer. Using the Galerkin method we obtain the spectrum of normal perturbations and find the critical values of the Grasshof number defining the boundary of the region of stability, relative to the parameters of the problem. We find, that for all orientations of the layer, the most "dangerous" (from the point of view of stability) are the monotonous perturbations with wavelengths of order of the layer thickness. It is interesting, that the transition from the thermal to the hydrodynamic instability on varying the orientation of the layer, is continuous. This is due to the change in the fundamental monotonous level of instability.

Convective motion in an inclined layer has a distinctive feature, namely the existence of the oscillatory instability parameters in the defined region. We find, that this oscillatory instability is associated (at the fundamental level) with short wavelength perturbations.

We note that an attempt of investigation of the convective motion in an inclined layer was performed earlier [5]. However, the decremental spectrum was not investigated and the only problem, was to obtain the critical Grasshof number. First approximations to the method utilized in [5] did not lead to the determination of the boundary of the monotonous stability over the whole range of the angles of inclination. Inferences concerning the oscillatory instability which appeared in [5] were also based on first approximations to the method and were not confirmed when higher order approximations were used [4].

1. Steady motion. Let us consider a plane fluid layer contained between two parallel planes $x = \pm h$ inclined at an angle α to the vertical (Fig. 1). Solid boundaries of the layer are kept at constant temperatures $\pm\theta$. Under these conditions the fluid cannot be in equilibrium (except in the limit when the layer is horizontal i.e. $\alpha = \pm 90°$) and convective motion takes place under any difference of temperatures.

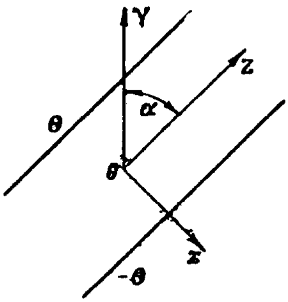

Fig. 1

Let us choose

$$h, \quad h^2/\nu, \quad g\beta\Theta h^2/\nu, \quad \Theta, \quad \rho g\beta\Theta h$$

as the units of distance, time, velocity, temperature and pressure respectively. Here ν denotes the kinematic viscosity ρ is the density, g is the acceleration due to gravity and β is the coefficient of thermal expansion.

Then the dimensionless equations of motion, heat conductivity and continuity, can be written as

$$\frac{\partial \mathbf{v}}{\partial t} + G(\mathbf{v}\nabla)\mathbf{v} = -\nabla p + \Delta\mathbf{v} + T\gamma \qquad (1.1)$$

$$\frac{\partial T}{\partial t} + G\mathbf{v}\nabla T = \frac{1}{P}\Delta T, \quad \operatorname{div}\mathbf{v} = 0 \qquad (1.2)$$

$$(G = g\beta\Theta h^3/\nu^2, \; P = \nu/\chi)$$

Dimensionless parameters G and P appearing in these equations, are the Grasshof and Prandtl numbers.

In the steady state the equations have a solution describing a plane parallel motion of fluid afar from the ends of the layer. The only velocity component different from zero is $v_z = v_0(x)$ and the temperature depends only on the transverse coordinate $T_0 = T_0(x)$. Then (1.1) and (1.2) yield the following expressions for the velocity, temperature and pressure in the steady flow

$$\frac{\partial p_0}{\partial z} = v_0'' + T_0\cos\alpha, \quad \frac{\partial p_0}{\partial x} = -T_0\sin\alpha, \quad T_0'' = 0 \qquad (1.3)$$

At the boundaries of the layer we have

$$v_0(\pm 1) = 0, \quad T_0(-1) = 1, \quad T_0(1) = -1 \qquad (1.4)$$

which, together with (1.3), gives

$$v_0 = {}^1/_6\,(x^3 - x)\cos\alpha, \quad T_0 = -x$$

$$p_0 = {}^1/_2 x^2 \sin\alpha + \text{const} \qquad (1.5)$$

Thus we see that in the steady state mode we have a flow with a cubic velocity profile, and a linear temperature distribution. The flow velocity is highest, when the layer is vertical ($\alpha = 0°$). As $\alpha \to \pm 90°$, the steady state flow is transformed into the equilibrium state of a horizontal layer heated from below (minus sign), or from above (plus sign).

2. Perturbation equations. Let $\mathbf{v}$, ϑ and p denote a small plane perturbation of a steady state flow (1.5). Eliminating from (1.1) the pressure by taking the curl and introducing the perturbation stream function $\psi(x, z, t)$ by

$$v_x = -\partial\psi/\partial z, \qquad v_z = \partial\psi/\partial x \qquad (2.1)$$

we obtain after the linearization, the following Eqs.:

$$\frac{\partial}{\partial t}\Delta\psi + G\left(v_0\frac{\partial}{\partial z}\Delta\psi - v_0''\frac{\partial\psi}{\partial z}\right) = \Delta\Delta\psi + \frac{\partial\vartheta}{\partial z}\sin\alpha + \frac{\partial\vartheta}{\partial x}\cos\alpha \qquad (2.2)$$

$$\frac{\partial\vartheta}{\partial t} + G\left(v_0\frac{\partial\vartheta}{\partial z} - T_0'\frac{\partial\psi}{\partial z}\right) = \frac{1}{P}\Delta\vartheta \qquad (2.3)$$

where Δ denotes a two-dimensional Laplacian in x and z. At the boundaries of the layer, the velocity and temperature perturbations disappear

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial z} = \vartheta = 0 \quad \text{when } x = \pm 1 \tag{2.4}$$

Eqs. (2.2) and (2.3) together with the boundary conditions (2.4), have solutions in the form of normal perturbations

$$\psi(x, z, t) = \Phi(x) e^{-\lambda t + ikz}, \qquad \vartheta(x, z, t) = T(x) e^{-\lambda t + ikz} \tag{2.5}$$

Here $\Phi(x)$ and $T(x)$ denote the perturbation amplitudes, λ is a complex decrement and k is a real wave number.

Inserting (2.5) into (2.2) and (2.3) we obtain a system of homogeneous linear equations for the perturbation amplitudes

$$\Delta\Delta\Phi + ikG\cos\alpha\, H\Phi + ikT\sin\alpha + T'\cos\alpha = -\lambda\Delta\Phi \tag{2.6}$$

$$P^{-1}\Delta T + ikG(T_0'\Phi - f_0 T\cos\alpha) = -\lambda T \tag{2.7}$$

where

$$H\Phi = f_0''\Phi - f_0\Delta\Phi, \qquad \Delta = \frac{\partial^2}{\partial x^2} - k^2, \qquad f_0 = {}^1/_6(x^3 - x)$$

Amplitudes Φ and T satisfy the homogeneous boundary conditions

$$\Phi = \Phi' = T = 0 \quad \text{when } x = \pm 1 \tag{2.8}$$

3. Method of solution. We shall use the Bubnov-Galerkin method to obtain the approximate solution of the boundary value problem (2.6) to (2.8).

We shall use the systems of perturbation amplitudes of the fluid at rest, as basis functions for approximating the perturbation amplitudes of the stream function $\Phi(x)$ and the temperature $T(x)$. These amplitudes are defined as the eigenfunctions of the following boundary value problems

$$\Delta\Delta\varphi_i = -\mu_i\Delta\varphi_i, \qquad \varphi_i(\pm 1) = \varphi_i'(\pm 1) = 0$$
$$P^{-1}\Delta\theta_k = -\nu_k\theta_k, \qquad \theta_k(\pm 1) = 0 \tag{3.1}$$

where μ_i and ν_k are the perturbation decrements in the fluid at rest.

Inserting the series

$$T = \sum_m \alpha_m\theta_m, \qquad \Phi = \sum_n \beta_n\varphi_n \tag{3.2}$$

into (2.6) and (2.7) and constructing the integral conditions of the Bubnov-Galerkin method we obtain a homogeneous linear system defining the coefficients α_m and β_n

$$\sum_m \left\{\left[ik\sin\alpha\frac{C_{mn}}{J_n} + \cos\alpha D_{mn}\right]\alpha_m + [ikG\cos\alpha H_{nm} + (\lambda - \mu_n)\delta_{mn}]\beta_m\right\} = 0$$

$$\sum_m \{[ikG\cos\alpha B_{lm} - (\lambda - \nu_m)\delta_{lm}]\alpha_m + ikGC_{lm}\beta_m\} = 0 \tag{3.3}$$

where

$$C_{mn} = \int_{-1}^{1}\theta_m\varphi_n dx, \qquad D_{mn} = \frac{1}{J_n}\int_{-1}^{1}\theta_m'\varphi_n dx, \qquad H_{nm} = \frac{1}{J_n}\int_{-1}^{1}\varphi_n H\varphi_m dx$$

$$B_{lm} = \int_{-1}^{1}\theta_l f_0\theta_m dx, \qquad J_n = \int_{-1}^{1}\varphi_n\Delta\varphi_n dx$$

Explicit expressions for the basis functions and for the matrix elements were given earlier in [3 and 4].

Condition of solvability of the homogeneous system (3.3) yields a characteristic equation which can be used to determine the decrements λ, the latter themselves being the eigenvalues of the boundary value problem (2.6) to (2.8). We had retained 8 to 12 terms in the expansions (3.2). Thus, to find the decrements λ we had to diagonalize matrices of the **16th** to the 24th order. This was performed on a digital computer, using the orthogonal step method [6].

4. Decremental spectrum and monotonous instability. Characteristic perturbation decrements λ depend on the parameters of the problem, namely, on the Grasshof (G) and the Prandtl (P) numbers, perturbation wave number k and the angle α of inclination of the layer to the vertical.

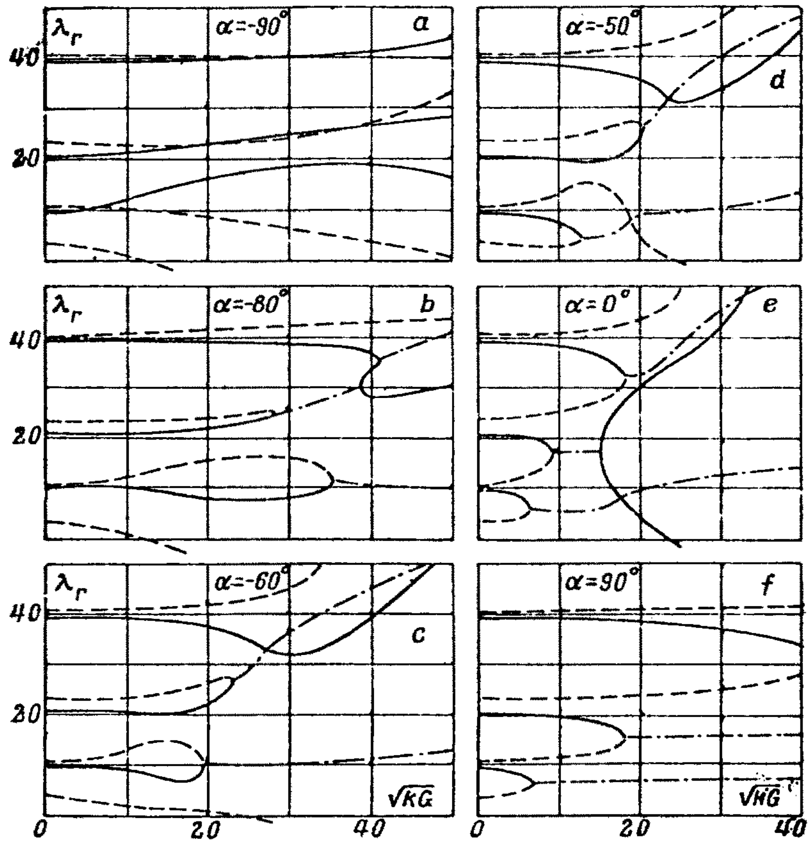

Fig. 2

Fig. 2 shows the lowest levels of the decremental spectrum as functions of the parameter $\varkappa = \sqrt{kG}$ for fixed values $k = 1$ and $P = 1$, and for various orientations of the layer. Solid lines represent the real levels of "isothermal" perturbations, while the broken lines refer to "nonisothermal" perturbations (for classification of the levels see [3]). Dash-dot lines represent real parts of the complex-conjugate level pairs.

Fig. 2a shows the decremental spectrum for a horizontal layer heated from below ($\alpha = -90°$, Rayleigh problem). Here all decrements are real (monotonous perturbations) and only simple intersections of levels are possible. Some of the decrements become negative with increasing $\varkappa$, thus giving rise to monotonous instability. Points of intersection of the decremental lines with the $\varkappa$-axis yield an increasing sequence of the critical Rayleigh numbers defining the spectrum of instability of the equilibrium of fluid relative to the appearance of convection (*).

(*Footnote carried forward to next page*)

When the orientation of the layer deviates from the horizontal, then the structure of the spectrum is altered (Fig. 2*b* and subsequent). Under an arbitrarily small deviation from the horizontal, simple intersections disappear. Real levels merge into complex conjugate pairs (oscillatory perturbations) which may sometimes (with increasing $\varkappa$) separate back into two real levels. Change of the structure of the spectrum is connected with the fact, that, when the layer is inclined when the developing perturbation pattern is superimposed on the steady convective flow, while in the limiting Rayleigh case ($\alpha = -90°$) they develop in the fluid at rest.

From the point of view of stability, the most interesting is the lowest real level intersecting the $\varkappa$-axis. Fig. 2 shows that the position of the critical point changes with inclination. Moreover, the interlocking of the levels results in a change of modes responsible for the onset of the monotonous instability. Thus, when $\alpha = -90°$, the instability depends on the lowest thermal level ν_0; when $\alpha = -50°$, the instability is generated by the level ν_1, while when $\alpha = 0°$, the critical point is given by the "mixture" of the thermal and isothermal levels ν_1 and μ_1.

Critical Grasshof Number G^* depends on the parameters k, P and α. When P and α are fixed, the function $G^*(k)$ defines a neutral curve which has a minimum at the point k_m. The corresponding minimum critical Grasshof number G^*_m defines the boundary of monotonous instability for the given values of P and α.

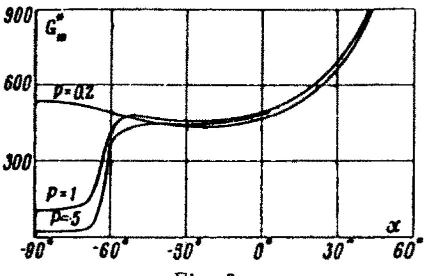

Fig. 3

Fig. 3 shows the dependence of G^*_m on α for several values of the Prandtl number, obtained from the spectra and from the neutral curves. Two ranges of values of the angle α can be distinguished. When the layer is nearly horizontal ($-90° < \alpha < -50°$), the critical values G^*_m depend essentially on the Prandtl number, and the product $G^*_m P$ is practically constant. Thus in this range of values of α, the boundary of instability is defined by the critical Rayleigh number $R = GP$ and this, as we know, is the feature characteristic of the thermal equilibrium instability. Indeed, within this range of α the breakdown of the steady state is caused by a Rayleigh type instability arising when the fluid is heated from below. When $\alpha = -90°$, a purely convective step appears. Over the range of angles close to $-90°$, the instability has also Rayleigh nature with one difference, namely, that it is developed over the pattern of a slow convective motion caused by a low horizontal temperature gradient.

In the range $\alpha > -50°$ the picture is different. Thermal instability mechanism is no longer predominant and it is absent when $\alpha > 0°$. This is due to the fact, that the heated layer is then situated in the upper part, and this corresponds to a stable temperature distribution. Under these conditions the breakdown of the steady state is caused by the hydrodynamic instability of two opposing convective flows. The Grasshof number G, practically independent of the Prandtl number, is now the defining parameter. The limiting case as $\alpha \to 90°$, corresponds to the horizontal layer heated from above. Since, as we know, a fluid heated from above is in a stable equilibrium, we have $G^* \to \infty$ as $\alpha \to 90°$.

The critical value of the wave number k_m corresponding to a minimum on the neutral curve is practically independent of P and weakly dependent on α. We find that, when α changes from $-90°$ to $+60°$, k_m decreases uniformly from 1.56 to 1.30.

It should be stressed that the changeover from the convective instability mechanism to the hydrodynamic one on increasing α from $-90°$ is smooth, since it corresponds to a continuous change of values along one branch of instability.

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*) We note that the critical values of the Grasshof number for the lowest four levels of instability, coincide to within a tenth of a percent with the values obtained in [7] from the exact characteristic equation for the Rayleigh spectrum.

5. Short wave perturbations and oscillatory instability. When a plane, horizontal fluid layer is heated from below, short wave perturbations lead to unstable equilibrium when the temperature difference is sufficiently high. We find that at high values of k, the critical Rayleigh numbers increase smoothly along the neutral stability curve $R(k)$ according to the law $R^* \sim k^4$[8].

If, on the other hand, the layer is vertical and heated from one side, then the steady motion is stable under the perturbations possessing high wave numbers: the neutral curve $G^*(k)$ has an asymptote at $k = k_0$. The limit value k_0 is almost independent of P and its numerical value is $k \approx 2$ (see [2 to 4]). All short wave perturbations with $k > k_0$, decay.

A question arises, concerning the behavior of the short wave perturbations with $k > k_0$ over the transitional range of angles. To answer this, we shall consider Fig. 4 showing two lowest levels of the decremental spectrum for $P = 1$ and $k = 2.5 > k_0$.

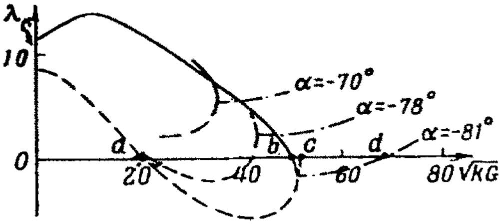

Fig. 4

At $\alpha = -81°$, both levels intersect the $\varkappa$-axis at the points a and b. At the point c real levels merge, forming a complex congate pair of decrements. The real part of the pair (common to both levels) is negative in the region (cd) and positive to the right of the point d. Thus, both perturbations decay monotonously in the region to the left of a. On the interval (ab) we have monotonous instability with respect to one of the perturbations, while on the interval (bc) both perturbations increase smooth. Interval (cd) represents the region of existence of the oscillatory instability in which both perturbations increase in the oscillatory manner. To the right of d we have a stable region (decaying oscillatory perturbations). Fig. 4 shows the lines representing two lowest decrements for $\alpha = -78°$ and $\alpha = -70°$, which illustrate the deformation of the spectrum with changing α.

Fig. 5 shows the stable region on the $(\sqrt{kG}, \alpha)$-plane for fixed $P = 1$ and $k = 2.5$. On the line $\alpha = -90°$ we have two lowest critical values of the Rayleigh instability spectrum. Broken vertical line indicates, for comparison, the case of $\alpha = -81°$ discussed above. Various regions are numbered as follows: *1* denotes the stable region (both perturbations decay smoothly); *2* denotes a stable region in which both perturbations decay in the oscillatory manner; *3* denotes the region of monotonous instability for the ν_0-perturbation; *4* denotes the region of monotonous instability for both perturbations and *5* the region of oscillatory instability (shaded).

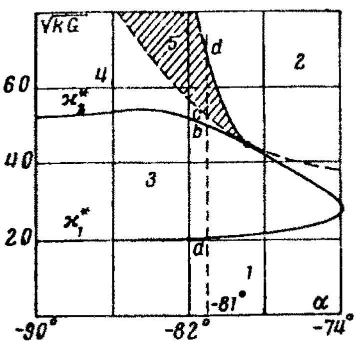

Fig. 5

Thus, we see that the instability under short wave perturbations occurs only when the layer is nearly horizontal, the position corresponding to the heating from below. With the angle of inclination to the horizontal increasing, the short wave instability disappears, and its rate of disappearance is proportional to the value of k. For $k = 2.5$, the perturbations begin to decay when the angle of inclination of the layer to the horizontal exceeds 15°.

An interesting fact which emerges from this is, that an oscillatory instability with respect to short wave perturbations exists in a certain parametric region. This does not take place in the limiting cases, when the layer is either horizontal, or vertical.

6. Upper levels of the instability spectrum. In conclusion we shall consider the problem of behavior of the upper instability levels with varying α. We know, that when the layer is horizontal and heated from below, then an infinite increasing sequence of the critical Grashof numbers exists for every value of the wave number. We have shown above that the lowest of these critical numbers is smoothly transformed, on varying α, into the critical value defining the boundary of stability of the steady convective flow. Since only one critical value and a unique level of instability exist (see [4]) when the layer is vertical, the question arises concerning the fate of the other levels of the Rayleigh instability spectrum when the orientation of the layer is varied.

To ascertain this problem, we have computed the decrements which gave, at $\alpha = -90°$, the second and third level of the Rayleigh instability spectrum. We found that when the layer deviates from the horizontal, then the upper instability levels close, the process being analogous to that described in Section 5.

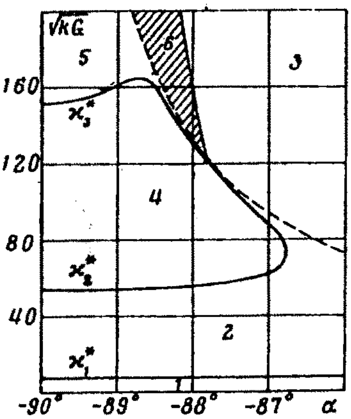

Fig. 6

Fig. 6 shows the stability pattern illustrating the behavior of the lowest three spectral modes (for $P = 1$ and $k = 1$). The regions are numbered as follows: *1* – stable region, decay of all three perturbations; *2* – smooth growth of the first and smooth decay of the second and third perturbations; *3* – smooth growth of the first, oscillatory decay of the second and third perturbations; *4* – smooth growth of the first and second and smooth decay of the third perturbation; *5* – the region of monotonous instability with respect to all three perturbations; *6* – region of oscillatory instability for the second and third perturbation.

Thus we see that, when the layer deviates from the horizontal by a small amount, the Rayleigh spectrum degenerates and only one (lowest) level of the monotonous instability is preserved.

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